https://www.linkedin.com/feed/update/urn:li:activity:6538449056312762368

Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{a+b+c} \geq 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \cdot \frac{1}{a^2 + b^2 + c^2}.$$

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Since $abc = 1$ implies $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = ab + bc + ca$,

$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = a^2b^2 + b^2c^2 + c^2a^2$ and $\frac{3}{a+b+c} = \frac{3abc}{a+b+c}$

then original inequality of the problem becomes

(**1**) $\quad ab + bc + ca - \frac{3abc}{a+b+c} \geq \frac{2(a^2b^2 + b^2c^2 + c^2a^2)}{a^2 + b^2 + c^2}.$

Assuming, due homogeneity of (**1**), that $a + b + c = 1$ and, denoting $p := ab + bc + ca,\ q := abc$, we obtain (**1**) in the form

$$p - 3q \geq \frac{2(p^2 - 2q)}{1 - 2p} \Leftrightarrow \frac{p(1 - 4p) + q(1 + 6p)}{1 - 2p} \geq 0.$$

Since $3p = ab + bc + ca \leq (a + b + c)^2 = 1$ and $9q \geq 4p - 1$ (Shure's Inequality $\sum a(a - b)(a - c) \geq 0$ in p, q notation and with normalization by $a + b + c = 1$) then $1 - 2p > 0$ and

$p(1 - 4p) + q(1 + 6p) \geq p(1 - 4p) + q_*(1 + 6p)$ where $q_* := \max\left\{0, \frac{4p - 1}{9}\right\}$.

For $p \in [1/4, 1/3]$ we have $p(1 - 4p) + q_*(1 + 6p) =$

$p(1 - 4p) + \frac{(4p - 1)(1 + 6p)}{9} = \frac{1}{9}(1 - 3p)(4p - 1) \geq 0$

and for $p \in (0, 1/4]$ we have $p(1 - 4p) + q_*(1 + 6p) = p(1 - 4p) \geq 0.$